Selected topics in probability

Thursday, 4 March 2021 13:03

Today: Random walks and electrical networks

Other possible topics: statistical physics

Galton-Watson trees, percolation, spin O(n)/Ising model, Random matrices, Boolean functions, Ergodic theorem, concentration inequalities.

Books: 1) Yvan Velenik / Chapitres choisis de théorie des probabilités.

2) Asaf Nachmias / Planar maps, random walks and circle packing.

3) Roman Vershynin / High-dimensional probability.

4) Geoffrey Grimmett / probability on graphs.

5) Levin, Peres, Wilmer / Markov chains and mixing times

⋮

Final grade: Based on homework assignments.

## Random walks and electrical networks

Chapter 2 in Nachmias book

Motivation: Lattice $\mathbb{Z}^d$ is the graph with vertices $\{(i_1, i_2, \ldots, i_d) : i_m \in \mathbb{Z}\}$ and $i \sim j$ if they differ by 1 in exactly one coordinate.

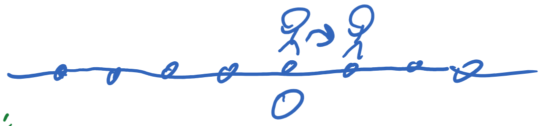
A random walker starts at $0 = (0, 0, \ldots, 0)$ and goes at each step to uniformly chosen neighbor

recurrent

goes at each step to
a uniformly chosen neighbor. recurrent
Is $\mathbb{P}(\exists n \geq 1, X_n = 0) = 1$ or $< 1$? ← transient

position of walker at time n

d = 1 : Recurrent.



i) Proof with martingales:
suppose walker starts at 1.
position $X_n$ of walker is a martingale.

$$\mathbb{E}(X_n | X_{n-1}) = X_{n-1}.$$

Define $T_0 := \min\{n \geq 0 : X_n = 0\}$ stopping time

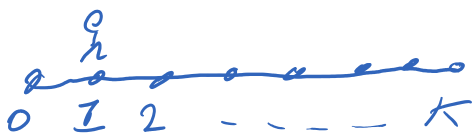
$M_n := X_{n \wedge T_0}$ (walker stopped at 0)
is also a martingale, which is non-negative.

Martingale convergence thm. (corollary):
A non-negative martingale converges almost surely.

Therefore $M_n \to L$ almost surely,
where $L$ is a random variable in $\mathbb{R}$.
since $M_n$ is integer-valued, it follows
that $M_n$ is constant from some n on,
therefore, it equals 0 from some n on.
That is $\mathbb{P}(T_0 < \infty) = 1$. finite almost surely

ii) Quantitative refinement: $T_{0,K} := \min\{n \geq 0 : X_n \in \{0, K\}\}$

$M_n' := X_{n \wedge T_{0,K}}$



is a bounded martingale.
Optional stopping thm.: $\mathbb{E}(M_{T_{0,K}}) = \mathbb{E} M_0 = 1$
since $\mathbb{E}(M_{T_{0,K}}) = \mathbb{P}(M_{T_{0,K}} = 0) \cdot 0 + \mathbb{P}(M_{T_{0,K}} = K) \cdot K$
$\to \mathbb{P}(M_{T_{0,K}} = K) = \frac{1}{K}$ (Gambler's ruin)

$$\Rightarrow P(M_{T_{0,k}} = k) = \frac{1}{k} \quad \text{(Gambler's ruin)}$$

iii) Another way: Start $(X_n)$ at $0$.

Set $N :=$ number of visits of $(X_n)$ to $0$ at times $n \geq 1$.

Since $(X_n)$ is a Markov Chain,

$N \sim \text{Geom}(P((X_n) \text{ never returns to } 0))$.

Therefore, $\mathbb{E}N = \dfrac{1}{P((X_n) \text{ never returns to } 0)}$,

But $\mathbb{E}N = \sum_{n=1}^{\infty} P(X_n = 0)$ $\quad (N = 1_{X_1=0} + 1_{X_2=0} + \dots)$

$X_n \sim 2\,\text{Bin}(n, \frac{1}{2}) - n \Rightarrow P(X_n = 0) = P(\text{Bin}(n, \frac{1}{2}) = \frac{n}{2})$

$$= \begin{cases} 0 & n \text{ odd} \\ 2^{-n}\binom{n}{n/2} & n \text{ even} \end{cases}$$

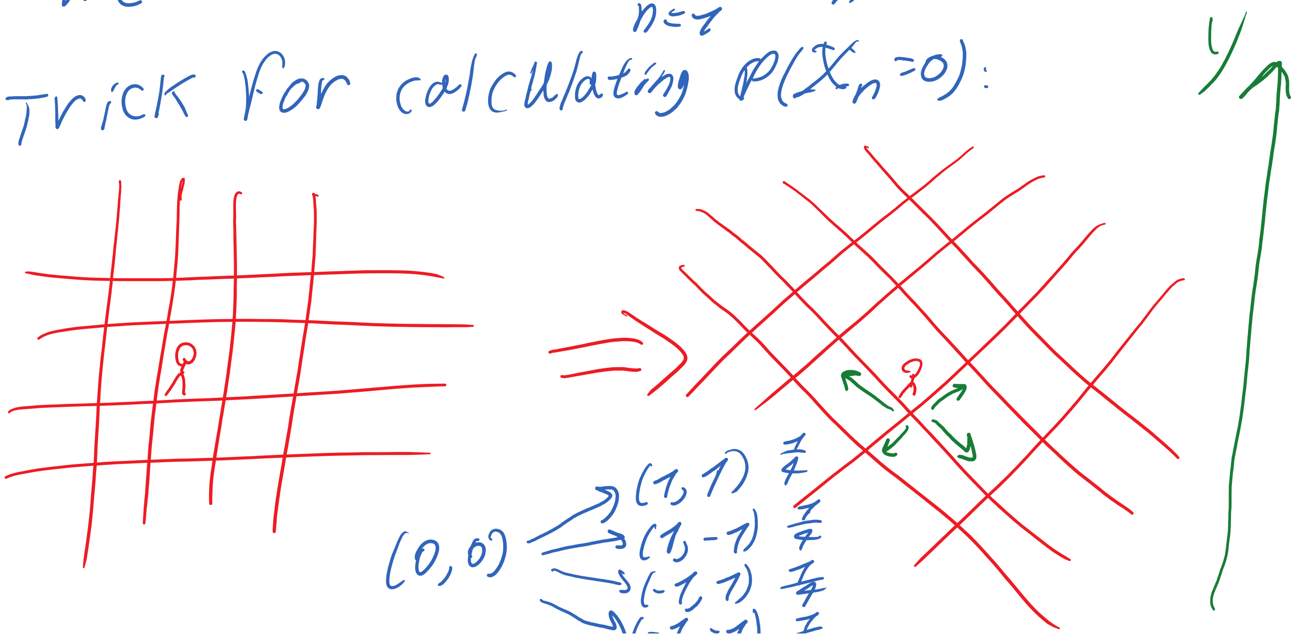
by stirling $\sim \frac{c}{\sqrt{n}}$

$\Rightarrow \mathbb{E}N = \infty$ since $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \infty$. Recurrent.

$d=2$: Recurrent.

We will show $\mathbb{E}N = \sum_{n=1}^{\infty} P(X_n = 0) = \infty$

Trick for calculating $P(X_n = 0)$:

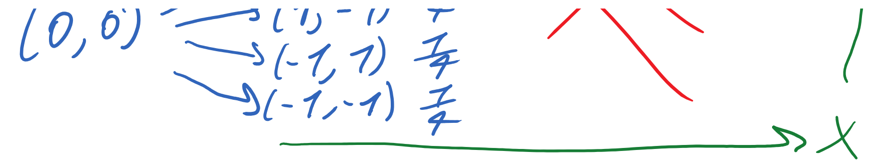



$(0,0) \to (1,1)$ $\frac{1}{4}$

$(0,0) \to (1,-1)$ $\frac{1}{4}$

$(0,0) \to (-1,1)$ $\frac{1}{4}$

$(0,0) \to (-1,-1)$ $\frac{1}{4}$




After rotating $45°$, writing

$$X_n = (X_{n,1}, X_{n,2})$$

x,y coordinates

Then $(X_{n,1})$, $(X_{n,2})$ are independent simple random walks in $d=1$.

Therefore $\mathbb{P}(X_n=0) = \mathbb{P}(X_{n,1}=0)^2 \sim \frac{c}{n}$ for even $n$, by prev. calculation.

$\Rightarrow \mathbb{E}N = \infty$ since $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

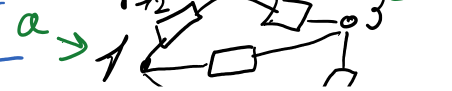
$d \geq 3$: Transient. Can do a similar calculation.

The methods used above for determining Recurrence/transience are not robust. Altering $\mathbb{Z}^d$ slightly (e.g., adding an edge) will make them fail. For this reason we now introduce a useful and non-trivial technology - electrical networks.

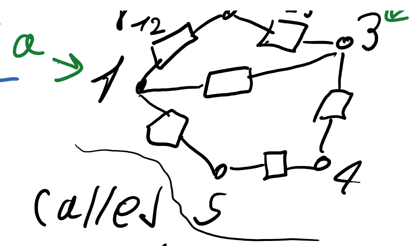
## Electrical networks

Def.: A network is a connected graph (self loops and multiple edges allowed) $G=(V,E)$ endowed with positive edge weights $(c_e)_{e \in E}$, called conductances. The reciprocals $r_e = \frac{1}{c_e}$ are called resistances.

## Harmonic Functions and Voltages



# Harmonic Functions and Voltages




We now focus on finite $G$.

**Def.:** A function $h: V \to \mathbb{R}$ is called harmonic at a vertex $x \in V$ if

$$h(x) = \frac{1}{\pi_x} \sum_{y : y \sim x} c_{xy} h(y)$$

we write $x \sim y$ to denote that $\{x, y\} \in E$

Where $\pi_x := \sum_{y : y \sim x} c_{xy}$.

Equivalently, $\sum_{y : y \sim x} c_{xy}(h(y) - h(x)) = 0$.

**Def.:** Given two distinct vertices $\{a, z\} \subseteq V$, a $h: V \to \mathbb{R}$ is called a Voltage if $h$ is harmonic at every $x \in V \setminus \{a, z\}$.

**Thm.:** Given a finite network $G = (V, E)$ with positive conductances $(c_e)_{e \in E}$ and two distinct vertices $\{a, z\} \subseteq V$, then for any $\alpha, \beta \in \mathbb{R}$ there exists a unique Voltage function $h$ s.t. $h(a) = \alpha$, $h(b) = \beta$.

**Lemma:** The space of voltages is a linear space.

**proof:** If $h_1, h_2$ harmonic at some $x \in V$ then also $h_1 + h_2$ and $c h_1$ for $c \in \mathbb{R}$ satisfy this.

**Lemma:** If $h: V \to \mathbb{R}$ is harmonic at all vertices then $h$ is constant.

**proof:** (maximum principle), set $M = \max_{x \in V} h(x)$.

set $A = \{x : h(x) = M\}$.

Set $A = \{x : h(x) = M\}$.
If $x \in A$ and $y \sim x$ then $y \in A$ by harmonicity of $h$ at $x$.
Since $G$ is connected conclude $A = V$.

**Lemma:** If $h : V \to \mathbb{R}$ is a voltage satisfying $h(a) = h(z) = 0$, then $h \equiv 0$.

**proof:** (again maximum principle)
Set $M = \max_{x \in V} h(x)$. Set $A = \{x \in V : h(x) = M\}$.
If $x \in A \setminus \{a, z\}$ and $y \sim x$ then $y \in A$ by harmonicity of $h$ at $x$.
Thus if $A \setminus \{a, z\} \neq \emptyset$ then also $a, z \in A$ by connectedness of $G$. Thus $M = 0$.
This is the case also if $a, z \in A$.
Therefore $M = 0$ so that $h \leq 0$.
By symmetry, $h \geq 0 \Rightarrow h \equiv 0$.

**Corollary:** If $h_1, h_2$ are voltages with $h_1(a) = h_2(a)$ and $h_1(z) = h_2(z)$ then $h_1 \equiv h_2$.

**proof:** previous lemma and linearity.

The uniqueness in the prev. corollary implies the existence of a voltage $h : V \to \mathbb{R}$ with given boundary values $h(a) = \alpha$, $h(b) = \beta$.
Indeed, such $h$ satisfies the $|V|$ equations

$$\left\{ \forall x \in V \setminus \{a, z\}, \quad \sum_{y : y \sim x} c_{xy} (h(y) - h(x)) = 0 \right.$$

$$\begin{cases} \forall x \in V \setminus \{a,z\} \ \ldots \sum_{y: y\sim x} \ldots \\ h(a)=\alpha \\ h(z)=\beta \end{cases}$$

The homogeneous system has a unique solution and therefore the non-homo. system has a solution.

We will show another proof, using random walks.

Define a random walk on $G$ by the transition prob.:

$$\mathbb{P}(X_n = y \mid X_{n-1}=x) = \begin{cases} \frac{c_{xy}}{\pi_x} & x\sim y \\ 0 & o/w \end{cases}$$

(Reminder: $\pi_x = \sum_{y: y\sim x} c_{xy}$)

Define the <u>hitting time</u>, for $x\in V$,

$$\tau_x := \min\{n\geq 0 : X_n = x\}.$$

Define $h(x) := \mathbb{P}_x(\tau_z < \tau_a)$.

$\mathbb{P}_x$ denotes $X_0 = x$

<u>Lemma:</u> $h$ is a voltage, $h(a)=0$, $h(z)=1$.

(This implies existance of voltages also for other boundary values, by adding a constant and multiplying by a constant).

<u>Proof:</u> It is clear that $h(a)=0$, $h(z)=1$.

$\forall x \in V\setminus\{a,z\}$, use the total prob. formula and Markov prop.:

$$h(x) = \mathbb{P}_x(\tau_z < \tau_a) = \mathbb{E}_x \mathbb{P}_x(\tau_z < \tau_a \mid X_1) =$$

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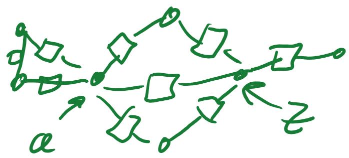
$$= \mathbb{E}_x h(X_1) = \sum_{y : y \sim x} \frac{c_{xy}}{\pi_x} h(y)$$

So that $h$ is harmonic at $x$.

**Corollary:** If $h$ is a voltage with $h(a) \le h(z)$ then

$$\forall x \in V, \quad h(a) \le h(x) \le h(z)$$

and also if $h(a) < h(z)$ and $x \in V \setminus \{a, z\}$ s.t. $x$ is in the conn. comp. of $z$ in $G \setminus \{a\}$ and also $x$ is in the conn. comp. of $a$ in $G \setminus \{z\}$ then




$$h(a) < h(x) < h(z).$$

**Proof:** Can assume $h(a) = 0$, $h(z) = 1$ by an affine transformation.

Then, by uniqueness, $h(x) = \mathbb{P}_x(\tau_z < \tau_a)$.

The corollary follows.

## Flows and currents

Flow = זרימה
Current = זרם
(זרם חשמלי)

Let $\vec{E}$ be the oriented edges of $G$ (each $e \in E$ appears in $\vec{E}$ with both orientations).

**Def:** A flow from $a$ to $z$ is a function $\theta : \vec{E} \to \mathbb{R}$ satisfying

1) antisymmetry: $\theta(xy) = -\theta(yx)$.

2.) Kirchhoff's node law:

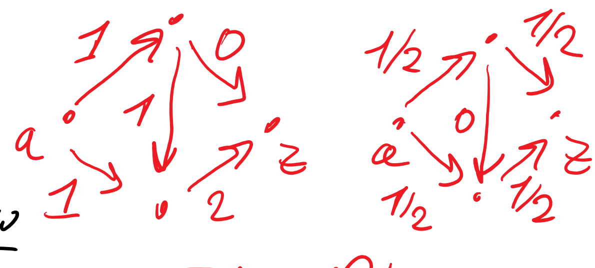
2) Kirchoff's node law:

$$\forall x \in V \setminus \{a,z\}, \sum_{y: y\sim x} \theta(xy) = 0$$

Lemma: The set of flows is a linear space.

Proof: clear.

Def.: Given a voltage $h: V \to \mathbb{R}$, define the current flow $\theta = \theta_h$ by $\theta(xy) = c_{xy}(h(y) - h(x))$.




Two flows

(Ohm's law: $V = I \cdot R \iff \theta(xy) \cdot r_{xy} = h(y) - h(x)$), $r_{xy} = 1/c_{xy}$

Remark: We choose the convention that the flow goes from low voltage to high voltage.

Lemma: The current flow is a flow.

Proof: antisymmetry is clear.

To check the node law, let $x \in V \setminus \{a,z\}$,

$$\sum_{y: y\sim x} \theta(xy) = \sum_{y: y\sim x} c_{xy}(h(y) - h(x)) = 0.$$

$$\boxed{\sum_{y: y\sim x} c_{xy}(h(y)-h(x)) = 0 \quad \text{Harmonicity at } x}$$

We now wish to understand when a flow $\theta$ is a current flow of some voltage.

Lemma: The current flow of some voltage satisfies Kirchoff's cycle law:

$\forall$ directed cycle $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_m$ in $G$,

$$\sum_{i=1}^{m} r_{e_i} \theta(\vec{e}_i) = 0$$

$=1$

$$\sum_{i=1} r_{e_i}\,\theta(\vec{e}_i) = 0$$

**Proof:** $\vec{e}_i := (x_{i-1}, x_i)$

$$\sum_{i=1}^{m} r_{e_i}\,\theta(\vec{e}_i) = \sum_{i=1}^{m} \underbrace{r_{e_i} c_{e_i}}_{=1} (h(x_i) - h(x_{i-1})) =$$

$$= \sum_{i=1}^{m} (h(x_i) - h(x_{i-1})) = 0 .$$

**Lemma:** If $\theta$ is a flow that satisfies Kirchoff's cycle law then $\exists$ voltage $h: V \to \mathbb{R}$ s.t. $\theta = \theta_h$ $(\theta(xy) = c_{xy}(h(y) - h(x)))$. Furthermore, $h$ is <u>unique</u> up to an additive constant.

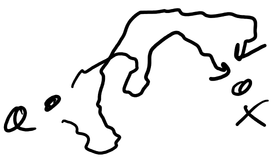
**Proof:** Set $h(a) = 0$.

Define for every $x \in V \setminus \{a\}$,

$$h(x) := \sum_{i=1}^{m} r_{e_i}\,\theta(\vec{e}_i)$$

where $\vec{e}_1, \text{—}, \vec{e}_m$ is a directed path from $a$ to $x$.

Kirchoff's cycle law ensures that $h(x)$ does not depend on the choice of path.




$\underline{\theta = \theta_h}$: That is $\forall x \sim y,\ \theta(xy) = c_{xy}(h(y) - h(x))$

This follows by taking a path $a \to y$ which is $a \to x, xy$ then

$$h(y) - h(x) = r_{xy}\,\theta(xy).$$

**$h$ is a Voltage:** Let $x \in V \setminus \{a, z\}$.

$$\sum_{y: y \sim x} c_{xy}(h(y) - h(x)) \underset{\theta = \theta_h}{=} \sum_{y: y \sim x} \theta(xy) \underset{\text{node law}}{=} 0$$

**Uniqueness:** If $\theta = \theta_{h_1} = \theta_{h_2}$ then

Uniqueness: If $\theta = \theta_{h_1} = \theta_{h_2}$ then

$$h_1(y) - h_1(x) = h_2(y) - h_2(x) \quad \forall x \sim y$$

$$\Rightarrow h_1(y) - h_2(y) = h_1(x) - h_2(x)$$

G is connected $\Rightarrow$ $h_1 \equiv h_2 + c$ for some $c \in \mathbb{R}$.

Def.: The strength of a flow $\theta$ from $a$ to $z$ is

$$\|\theta\| := \sum_{x : a \sim x} \theta_{ax}$$

(that is, the flow leaving $a$).

Lemma: $\|\theta\| = \sum_{x : x \sim z} \theta_{xz}$.

Proof: $0 \overset{\text{antisymmetry}}{=} \sum_{x \in V} \sum_{y : y \sim x} \theta_{xy} =$

$$= \sum_{x : x \sim a} \theta_{ax} + \sum_{x : x \sim z} \theta_{zx} + \sum_{x \in V \setminus \{a, z\}} \underbrace{\sum_{y : y \sim x} \theta_{xy}}_{= 0 \text{ by node law}}$$

$$\Rightarrow \|\theta\| = \sum_{x : x \sim a} \theta_{ax} \overset{\text{antisymmetry}}{=} \sum_{x : x \sim z} \theta_{xz}$$

We now discuss uniqueness of flows.

Lemma: If $\theta_1, \theta_2$ are flows satisfying the cycle law and $\|\theta_1\| = \|\theta_2\|$ then $\theta_1 = \theta_2$.

Proof: Since $\theta_1, \theta_2$ satisfy the cycle law then $\theta_1 = \theta_2$

**Proof:** Since $\theta_1, \theta_2$ satisfy the cycle law then $\theta_1 = \theta_{1,h_1}$, $\theta_2 = \theta_{2,h_2}$ for some voltages $h_1, h_2$.

Let $\bar{\theta} := \theta_1 - \theta_2$. By linearity,

$$\bar{\theta} = \bar{\theta}_{h_1 - h_2}.$$

We will show that $h_1 = h_2 + c$ for some $c \in \mathbb{R}$ and therefore $\bar{\theta} = \bar{\theta}_c = 0$.

To this end, check that $h_1 - h_2$ is harmonic at all vertices of $V$.

We need only check harmonicity at $a$ and $z$ and this follows by a simple calculation using the fact that $\|\theta_1\| = \|\theta_2\|$.

**Def.:** The unit current flow $\theta$ is the unique flow satisfying the cycle law with $\|\theta\| = 1$.